The Decidable and the Undecidable. A Survey of Recent Results

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Plan for the talk

- (Un)decidability: what and why?
- Propositional team logics and their decidability
- Exploring boundaries between the decidable and the undecidable
    - Solving problems and obtaining insights along the way
    - Using insights to solve one last problem

(Un)decidability: what and why?

What?

A decision problem is a collection of inputs I, with a yes-or-no question for each $i \in I$.

A decision problem is decidable if there is an effective method that, given any $i \in I$, accurately answers the question. Otherwise, it is undecidable.

A logic $\mathbf{L}$, in a language $\mathcal{L}$, is decidable if there is an effective method that, given any $\varphi \in \mathcal{L}$, determines whether $\mathbf{L} \vdash \varphi$. Otherwise, it is undecidable.

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Why? *Because it is a deep, profound and significant conceptual distinction.*

Propositional team logics and their decidability

Traditionally (in, e.g., CPC), formulas φ are evaluated at single valuations $v : \mathbf{Prop} \to \{0, 1\}$,

$$v \models \varphi.$$

In team semantics, formulas φ are evaluated at sets ('teams') of valuations $s \subseteq \{v \mid v : \mathbf{Prop} \to \{0, 1\}\}$,

$$s \models \varphi.$$

Definition (some team-semantic clauses)

Let $X := \{v \mid v : \mathbf{Prop} \to \{0, 1\}\}$. For $s \in \mathcal{P}(X)$, we define

$s \models p$	iff	$\forall v \in s : v(p) = 1$,
$s \models \varphi \wedge \psi$	iff	$s \models \varphi$ *and* $s \models \psi$,
$s \models \varphi \mathbin{\rotatebox[origin=c]{180}{$\mathbb{A}$}} \psi$	iff	$s \models \varphi$ *or* $s \models \psi$,
$s \models \sim\varphi$	iff	$s \not\models \varphi$,
$s \models \varphi \vee \psi$	iff	there exist $s', s'' \in \mathcal{P}(X)$ such that $s' \models \varphi$; $s'' \models \psi$; and $s = s' \cup s''$.

Observation. All propositional team logics are decidable: given φ, simply check whether $s \models \varphi$ for all $s \subseteq \{v \mid v : \mathbf{Prop}(\varphi) \to \{0, 1\}\}$.

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Yet, this explanation is hardly satisfactory.
What is it that makes propositional team logics
decidable, *and others not?*

Team semantics as relational semantics

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In fact, if we take all powerset frames $(\mathcal{P}(X), \cup)$, redefine the base clause

$$(\mathcal{P}(X), \cup, V), s \Vdash p \qquad \text{iff} \qquad s \in V(p),$$

and only allow principal valuations $V : \mathbf{Prop} \to \{{\downarrow}s \mid s \in \mathcal{P}(X)\}$, we get sound and complete relational semantics for team logics.

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This induces a powerset frame $\mathbb{F} = (\mathcal{P}(X), \cup)$, where '$\circ$' is a binary modality referring to the ternary $\cup$-relation $s = s' \cup s''$; and a model $\mathbb{M} = (\mathcal{P}(X), \cup, V)$ with a 'principal valuation', i.e.,

$$V(p) := \{s \in \mathcal{P}(X) \mid \forall v \in s : v(p) = 1\} = {\downarrow}\{v \in X \mid v(p) = 1\}.$$

In fact, if we take all powerset frames $(\mathcal{P}(X), \cup)$, redefine the base clause

$$(\mathcal{P}(X), \cup, V), s \Vdash p \qquad \text{iff} \qquad s \in V(p),$$

and only allow principal valuations $V : \mathbf{Prop} \to \{{\downarrow}s \mid s \in \mathcal{P}(X)\}$, we get sound and complete relational semantics for team logics.

Proof. A simple p-morphism argument.

Powerset frames and Boolean frames

Summarizing, (i) team logics are decidable, and (ii) relational semantics for team logics are given by powerset frames $(\mathcal{P}(X), \cup)$ with principal valuations $V : \mathbf{Prop} \to \{\downarrow s \mid s \in \mathcal{P}(X)\}$.

Question: *Sticking with the signature* $\{\wedge, \vee, \sim, \circ\}$*, what happens if we allow for* *arbitrary* *valuations* $V : \mathbf{Prop} \to \mathcal{P}\mathcal{P}(X)$*? Does the logic remain decidable?*

In fact, this question is intimately related with an open problem: Goranko and Vakarelov (1999) consider the logic of Boolean frames – instead of a powerset $\mathcal{P}(X)$, the carrier is a Boolean algebra B – and raises the problem of its decidability.[1]

Theorem

The logic of powerset frames, in the signature $\{\wedge, \vee, \sim, \circ\}$, with arbitrary valuations is *undecidable*. And so is the hyperboolean modal logic of Goranko and Vakarelov (1999).

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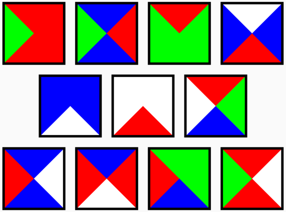
Proof method: tiling

- A (Wang) tile is a square with colors on each side.
- The tiling problem: given any finite set of tiles $\mathcal{W}$, determine whether each point in the quadrant $\mathbb{N}^2$ can be assigned a tile from $\mathcal{W}$ such that neighboring tiles share matching colors on connecting sides.
- The tiling problem was introduced by Wang (1963) and proven undecidable by Berger (1966).



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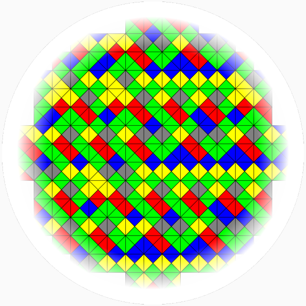

Figure 2: A tiling of the plane

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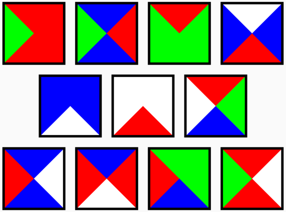

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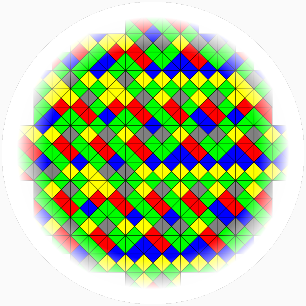

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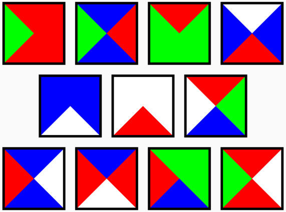

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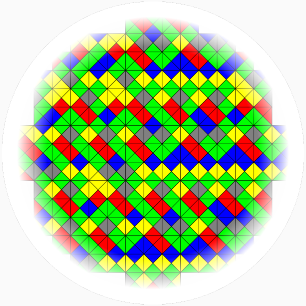

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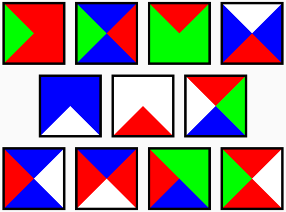

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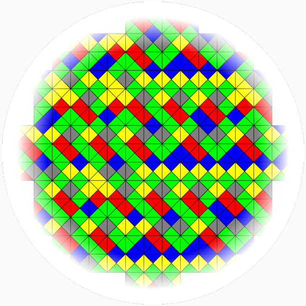

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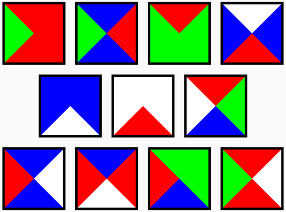

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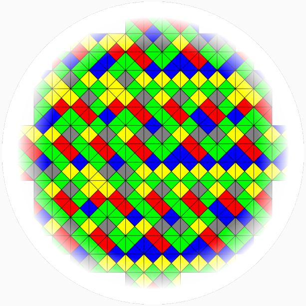

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Proof idea.

For each finite set of tiles $\mathcal{W}$, we construct a formula $\phi_{\mathcal{W}}$ such that $\mathcal{W}$ tiles the quadrant if and only if $\phi_{\mathcal{W}}$ is satisfiable. □

Dividing the proof into two lemmas, corresponding to a direction each, we can prove both results in one go:

Lemma

If $\phi_{\mathcal{W}}$ is satisfiable (in a Boolean frame), then $\mathcal{W}$ tiles $\mathbb{N}^2$.

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Insight 1: valuations matter

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Question: *Since we can weaken from powersets to Boolean algebras and stay undecidable, how much further can we go while remaining undecidable?*

Weakening from powersets $(\mathcal{P}(X), \cup)$ to general (join-)semilattices $(S, \sqcup)$, we get a the problem posed by Bergman (2018) and Jipsen et al. (2021) (and SBK (2023a)).

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Question: *What if we weaken even further than semilattices?*

(Partial) answer: As semilattices are partial orders '$\leq$' with all binary suprema, we could consider the logic of all *partial orders simpliciter*. This is modal information logic, which is proven decidable in SBK (2023b).

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Answer: If we stick to semilattices but *omit negation*, so signature is $\{\wedge, \vee, \circ\}$, we obtain *Finean truthmaker semantics*, proven decidable in SBK (2023a).

Semilattice frames, associativity and negation

Question: *Since we can weaken from powersets to Boolean algebras and stay undecidable, how much further can we go while remaining undecidable?*

Weakening from powersets $(\mathcal{P}(X), \cup)$ to general (join-)semilattices $(S, \sqcup)$, we get a the problem posed by Bergman (2018) and Jipsen et al. (2021) (and SBK (2023a)).

Theorem

For any class $\mathcal{S}$ of semilattices containing $(\mathcal{P}(\mathbb{N}), \cup)$, its logic in the signature $\{\wedge, \vee, \sim, \circ\}$, is undecidable.

Question: *What if we weaken even further than semilattices?*

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Insight 2: associativity matters

Insight 3: negation matters

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Theorem: $\mathbb{S}$ is undecidable

References I

Berger, R. (1966). *The undecidability of the domino problem.* English. Vol. 66. Mem. Am. Math. Soc. Providence, RI: American Mathematical Society (AMS). DOI: 10.1090/memo/0066 (cit. on pp. 33–37).

Bergman, C. (2018). "Introducing Boolean Semilattices". In: *Don Pigozzi on Abstract Algebraic Logic, Universal Algebra, and Computer Science. Outstanding Contributions to Logic.* Ed. by J. Czelakowski. Springer, pp. 103–130 (cit. on pp. 43–54).

Goranko, V. and D. Vakarelov (1999). "Hyperboolean Algebras and Hyperboolean Modal Logic". In: *Journal of Applied Non-Classical Logics* 9.2-3, pp. 345–368. DOI: 10.1080/11663081.1999.10510971 (cit. on pp. 27–32, 38–41).

References II

- Jipsen, P., M. Eyad Kurd-Misto, and J. Wimberley (2021). **"On the Representation of Boolean Magmas and Boolean Semilattices".** In: *Hajnal Andréka and István Németi on Unity of Science: From Computing to Relativity Theory Through Algebraic Logic.* Ed. by J. Madarász and G. Székely. Cham: Springer International Publishing, pp. 289–312. DOI: `10.1007/978-3-030-64187-0_12` (cit. on pp. 43–54).
- Knudstorp, S. B. (2023a). **"Logics of Truthmaker Semantics: Comparison, Compactness and Decidability".** In: *Synthese* (cit. on pp. 43–54).
- — (2023b). **"Modal Information Logics: Axiomatizations and Decidability".** In: *Journal of Philosophical Logic* (cit. on pp. 43–54).
- Urquhart, A. (1984). **"The undecidability of entailment and relevant implication".** In: *Journal of Symbolic Logic* 49, pp. 1059–1073 (cit. on pp. 57–74).

References III

Urquhart, A. (2016). **"Relevance Logic: Problems Open and Closed".** In: *The Australasian Journal of Logic* 13 (cit. on pp. 57–74).

Wang, H. (1963). **"Dominoes and the $\forall\exists\forall$ case of the decision problem".** In: *Mathematical Theory of Automata*, pp. 23–55 (cit. on pp. 33–37).

Thank you!